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A note on local spectra and multicyclic hyponormal operators

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0. Introduction. For a compact subset E of the complex plane, $R(E)$ denotes the set of rational functions with poles off E. An operator A acting on a Hilbert space $\mathfrak{H}$ is said to be n-multicyclic if there are n vectors $g_1, \ldots, g_n \in \mathfrak{H}$, called generating vectors, such that $\mathfrak{H} = \bigvee \{r(A)g_i \colon r \in R(\sigma(A)),\ 1 \leq i \leq n\}$. The following theorem of BERGER and SHAW [1] is very well known.

Theorem A. *Let $A \in \mathcal{B}(\mathfrak{H})$ be hyponormal, with n-multicyclic generating vectors $g_1, \ldots, g_n$. Then*

$$\operatorname{tr}[A^*, A] \leq (n/\pi)\omega(\sigma(A)),$$

where $[A^*, A] = A^*A - AA^*$, and ω denotes the planar Lebesgue measure.

The purpose of this paper is to sharpen this theorem as follows:

Main Theorem. *Let $A \in \mathcal{B}(\mathfrak{H})$ be hyponormal, with n-multicyclic generating vectors $g_1, \ldots, g_n$. Then*

$$\operatorname{tr}[A^*, A] \leq (1/\pi)\left[\omega(\sigma_A(g_1)) + \ldots + \omega(\sigma_A(g_n))\right],$$

where $\sigma_A(g_i)$, $i = 1, 2, \ldots, n$, are local spectra of A.

This formulation is due to the consideration of the operator $A = T_z \oplus T_{z/2}$ defined on $H^2(\chi_{\mathbf{D}}\omega) \oplus H^2(\chi_{\mathbf{D}}\omega)$ by multiplication by z and $z/2$ respectively, where $\mathbf{D}$ is the unit disk. It is clear that A is a 2-multicyclic hyponormal operator, with generating vectors $g_1 = 1 \oplus 0$ and $g_2 = 0 \oplus 1$, and

$$\operatorname{tr}[A^*, A] = (1/\pi)[\omega(\mathbf{D}) + \omega(\mathbf{D})/4] = (1/\pi)\left[\omega(\sigma_A(g_1)) + \omega(\sigma_A(g_2))\right].$$

This shows that our Main Theorem is sharper then Theorem A. As for the proof, it is carried out by "localizing" that given in [1].

Remark. In [5], D. VOICULESCU has extended Theorem A to cover also operators whose self-commutators possess trace-class negative parts. Since these oper-

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ators may not satisfy property (C) (defined below) even when they are cyclic (sample: the backward shift), it seems to be difficult to sharpen this generalized version according to our scheme.

Throughout this paper, all operators are bounded, acting on complex separable Hilbert space of infinite dimension.

1. Preliminaries. The following notions and lemmas come from Dunford and Schwartz [2], p. 2171.

Definition. Let $A\in\mathcal{B}(\mathfrak{K})$. For each $x\in\mathfrak{K}$ the symbol $[x]$ will be used for the closed linear manifold spanned by all vectors $(\lambda I-A)^{-1}x$ with $\lambda\in\varrho(A)$; $\mathfrak{M}(\sigma)$ denotes the set of all x whose spectrum is contained in the set σ: $\sigma_A(x)\subset\sigma$.

Note here that if $A\in\mathcal{B}(\mathfrak{K})$ is an n-multicyclic operator, with generating vectors $g_1, \dots, g_n$, then $\mathfrak{K}=[g_1]\vee\dots\vee[g_n]$.

Lemma A. *$x\in[x]$ and $f(A)[x]\subset[x]$ for $f\in F(\sigma(A))$, where $F(\sigma(A))$ denotes the set of all complex functions which are single valued and analytic on an open set containing $\sigma(A)$.*

Lemma B. *If A has property* (C) *(i.e., $\mathfrak{M}(\sigma)$ is closed when σ is closed), then for $x\in\mathfrak{K}$ we have $\sigma(A|_{[x]})=\sigma_A(x)$, the local spectrum of A at x.*

The next theorem is due to Stampfli [4] for $\sigma(A)=\sigma_C(A)$, the continuous spectrum of A; Radjabalipour [3] put the finishing touch by showing that it remains valid for $\sigma(A)\neq\sigma_C(A)$.

Theorem B. *If A is a hyponormal operator then A satisfies property* (C).

Combining Lemma B and Theorem B, one sees immediately that if A is hyponormal then $\sigma(A|_{[x]})=\sigma_A(x)$. This observation makes possible the "localization" of the Subspace Dominance Lemma of Berger and Shaw [1]. Indeed, due to the observation, it makes sense to introduce the following notation for hyponormal operators:

$$[x; A', E] = \vee\{(\lambda I-A')^{-1}x \mid \lambda\notin E\},$$

where $A'=A|_{[x]}$ and $E\supset\sigma_A(x)$ $(=\sigma(A'))$. At the same time, it is crucial to notice that $[x]=[x; A', \sigma(A')]$. (Proof: $[x]\supset[x; A', \sigma(A')]$ is obvious since A' is an operator from $[x]$ to $[x]$. The reverse inclusion can be established by observing that $x\in[x]$ and $(\lambda I-A)^{-1}x=(\lambda I-A')^{-1}x$ for all $\lambda\in\varrho(A)$.)

To end this section, we list lemmas from [1], which are needed in the proof of the Main Theorem.

Structure Lemma. *Let T and A be hyponormal operators on $\mathfrak{H}$ and $\mathfrak{K}$ respectively, and let W: $\mathfrak{H}\to\mathfrak{K}$ be a trace class operator with dense range, such that $WT=AW$. Then* $\operatorname{tr}[A^*, A]\leq\operatorname{tr}[T^*, T]$.

Intertwining Lemma. *Let (U, k_z, x) be an analytic evaluation for $T\in\mathcal{B}(\mathfrak{H})$ and suppose that x is a 1-multicyclic vector for T. If $u\in\mathfrak{H}$, let $\hat{u}(z)=(u, k_z)$, for $z\in U$. Let $A\in\mathcal{B}(\mathfrak{K})$ such that $\sigma(A)\subset U$ and let $y\in\mathfrak{K}$. Define $W\colon \mathfrak{H}\to\mathfrak{K}$, $Wu=\hat{u}(A)y$. The $WT=AW$ and W lies in trace class.*

For convenience, we copy the definition of analytic evaluation here from [1]. Let $T\in\mathcal{B}(\mathfrak{H})$. Suppose there is a map $z\mapsto k_z$, from the open set U to $\mathfrak{H}$, which is conjugate analytic as a map into $\mathfrak{H}$ in the strong topology, and such that there is a vector $x\in\mathfrak{H}$ satisfying $\langle r(T)x, k_z\rangle=r(z)$ for all rational functions with poles off $\sigma(T)$, and all $z\in U$. Then the triple (U, k_z, x) will be called an analytic evaluation for T, if $T^*k_z=\bar{z}k_z$ for all $z\in U$.

Second Computational Lemma. *Let $U_1, \ldots, U_n$ be open sets with disjoint closures, each bounded by finitely many disjoint smooth Jordan curves. Let $U=\bigcup_{i=1}^{n} U_i$ and $\mathfrak{H}=R^2(\chi_U\text{-}\omega)$ (the closure of $R(\chi_U\text{-})$ in $L^2(\chi_U\text{-}\omega)$). Then T_z on $\mathfrak{H}$ satisfies* $\operatorname{tr}[T_z^*, T_z]\leq\pi^{-1}\omega(U)$.

2. Proof of the main theorem. To start with, it is necessary to "localize" the Subspace Dominant Lemma in [1].

Lemma. *Let $A\in\mathcal{B}(\mathfrak{H})$ be an n-multicyclic hyponormal operator, with generating vectors $g_1, \ldots, g_n$. Thus $\mathfrak{H}=[g_1]\vee\ldots\vee[g_n]=[g_1;\, A_1, \sigma(A_1)]\vee\ldots\vee[g_n;\, A_n, \sigma(A_n)]$, where $A_i=A|_{[g_i]}$. Let E_i be a compact set containing $\sigma_A(g_i)$ $(=\sigma(A_i))$ for $i=1, 2, \ldots, n$, and let $\mathfrak{B}=[g_1\colon A_1, E_1]\vee\ldots\vee[g_n;\, A_n, E_n]$. Then $\mathfrak{B}$ is an invariant subspace for A, $A|_{\mathfrak{B}}$ is hyponormal, $\sigma(A_i|_{[g_i;\, A_i, E_i]})\subset E_i$ for $i=1, 2, \ldots, n$, $A|_{\mathfrak{B}}$ is n-multicyclic with generating vectors $g_1, g_2, \ldots, g_n$ and* $\operatorname{tr}[A^*, A]\leq\operatorname{tr}[(A|_{\mathfrak{B}})^*, A|_{\mathfrak{B}}]$.

Proof. Assume $\operatorname{tr}[(A|_{\mathfrak{B}})^*, A|_{\mathfrak{B}}]<\infty$. Let $\{a_{ij}\}_{j=1}^{\infty}$ be a sequence of points in $E_i-\sigma_A(g_i)$ which land densely in each component of $\sigma_A(g_i)^C$ which lies entirely in E_i. Let

$$r_{im}(z) = \prod_{j=1}^{m} (z-a_{ij})^{-1}.$$

Let $\mathfrak{B}_{im}=r_{im}(A_i)[g_i;\, A_i, E_i]$, $\mathfrak{B}_{i0}=[g_i;\, A_i, E_i]$ and let $\mathfrak{B}_m=\bigvee_{i=1}^{n}\mathfrak{B}_{im}$. Clearly $\mathfrak{B}_{m+1}\supset\mathfrak{B}_m$, $\operatorname{rank}(\mathfrak{B}_{m+1}-\mathfrak{B}_m)\leq n$ and $\mathfrak{B}_m\uparrow\mathfrak{H}$ strongly. The rest of the proof is identical to that of Berger and Shaw's and thus omitted.

Proof of the Main Theorem. Let U_i, for $i=1, \ldots, n$, be open sets bounded by a finite number of disjoint smooth Jordan curves such that $\sigma_A(g_i)\subset U_i$ and $\omega(U_i)-\omega(\sigma_A(g_i))$, $i=1, \ldots, n$, are small. Let $\mathfrak{K}'$ be the subspace spanned by $[g_1;\, A_1, U_1^-], \ldots, [g_n;\, A_n, U_n^-]$. Let $A'=A|_{\mathfrak{K}'}$. A' is hyponormal, $\sigma(A_i|_{[g_i, A_i, U_i^-]})\subset U_i$ for $i=1, \ldots, n$, and $\{g_i\}$ is a set of n-multicyclic vectors for A'. Now let

$T=\sum_{i=1}^{n}\oplus T_z$ acting on $\mathfrak{H}=\sum_{i=1}^{n}\oplus R^2(\chi_{U_i^-}\omega)$. It is enough to establish:

$$\operatorname{tr}[A^*, A] \leqq \operatorname{tr}[A'^*, A'] \leqq \operatorname{tr}(T^*, T] \leqq (1/\pi)[\omega(U_1)+\ldots+\omega(U_n)].$$

The first and the third inequalities are due to the "local" subspace dominance lemma and the Second Computational Lemma, respectively. The second inequality can be claimed by producing an intertwining map between T and A' satisfying the conditions of the Structure Lemma.

$R^2(\chi_{U_i^-}\omega)$ has reproducing kernel k_z at each $z\in U_i$, for $i=1, \ldots, n$. The maps $z\mapsto k_z$ are strongly conjugate analytic, and the triples $(U_i, k_z, 1)$, $i=1, \ldots, n$, are analytic evaluations. Thus the map W_i: $R^2(\chi_{U_i^-}\omega)\to[g_i; A_i, U_i^-]$ defined by $W_i f=\hat{f}(A_i')g_i$ lies in trace class and $W_i T_z=A_i' W_i$ where $A_i'=A_i|_{[g_i;\, A_i,\, U_i^-]}$. Define W: $\sum\oplus R^2(\chi_{U_i^-}\omega)\to K'$ by $W=\sum W_i$. W lies in trace class and $WT=A'W$. Indeed,

$$WT(f_1\oplus\ldots\oplus f_n)=W(T_z f_1\oplus\ldots\oplus T_z f_n)=A_1'\hat{f}_1(A_1')g_1+\ldots+A_n'\hat{f}_n(A_n')g_n,$$

$$A'W(f_1\oplus\ldots\oplus f_n)=A'[\hat{f}_1(A_1')g_1+\ldots+\hat{f}_n(A_n')g_n]=A_1'\hat{f}_1(A_1')g_1+\ldots+A_n'\hat{f}_n(A_n')g_n.$$

The last equality holds because $\hat{f}_i(A_i')g_i\in[g_i;\ A_i,\ U_i^-]$ for $i=1, \ldots, n$. Clearly the range of W is dense in $\mathfrak{K}'$. The proof is complete.

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